Dyson-Index-Like Behavior of Bures Separability Functions

Paul B. Slater[*]

ISBER, University of California,

Santa Barbara, CA 93106

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Abstract

We conduct a study based on the Bures (minimal monotone) metric, analogous to that recently reported for the Hilbert-Schmidt (flat or Euclidean) metric (arXiv:0704.3723v2). Among the interesting results obtained there had been proportionalities—in *exact* correspondence to the Dyson indices $\beta = 1, 2, 4$ of random matrix theory—between the fourth, second and first powers of the separability functions $\mathcal{S}_{type}(\mu)$ for real, complex and quaternionic qubit-qubit scenarios, Here $\mu = \sqrt{\frac{\rho_{11}\rho_{44}}{\rho_{22}\rho_{33}}}$, with ρ being a 4×4 density matrix. Separability functions have proved useful—in the framework of the Bloore (correlation coefficient/off-diagonal scaling) parameterization of density matrices—for the calculation of separability *probabilities*. We find—for certain, basic simple scenarios (in which the diagonal entries of ρ are unrestricted, and one or two off-diagonal [real, complex or quaternionic] pairs of entries are nonzero) —that these proportionalities no longer strictly hold in the Bures case, but do come remarkably close to holding.

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[*]Electronic address: slater@kitp.ucsb.edu

In our recent study [1], we reported a number of developments of value in resolving the clearly challenging and conceptually important question of the probabilities—using the measure induced by the Hilbert-Schmdit (HS) metric—that generic real, complex and quaternionic qubit-qubit and qubit-qutrit states are separable/disentangled. An essential component in this progress was the use of the (simple) parameterization of the $n \times n$ density matrices (ρ) that had been originally proposed by Bloore [2]. This involves reparameterizing the off-diagonal entries ρ_{ij} as $\sqrt{\rho_{ii}\rho_{jj}}(x_{ij}+\mathbf{i}y_{ij})$. In the real case ($y_{ij}=0$), the $n \times n$ matrix of x_{ij}'s, being necessarily nonnegative definite, with $x_{ij} \in [-1, 1]$, has the form of a *correlation matrix*—a basic object of study in descriptive statistics [3, 4, 5]. (Correlation matrices can be obtained by standardizing *covariance* matrices. Density matrices have been viewed as covariance matrices of multivariate normal [Gaussian] distributions [6]. The possible states of polarization of a two-photon system are describable by six Stokes parameters and a 3×3 "polarization correlation" matrix [7].)

A major virtue (of course, unrecognized more than thirty years ago in the 1976 paper [2]) of the Bloore parameterization is that it allows one to express the well-known Peres-Horodecki positive-partial-transposition criterion [8, 9, 10] for separability using fewer variables than one would naively anticipate. Since we are largely concerned with the evaluation of high-dimensional integrals, this reduction in number of relevant variables is certainly of considerable importance.

Here, we parallel the sequential approach of Życzkowski and Sommers in that they, first, computed the *total* volume of (separable *and* nonseparable) $n \times n$ density matrices in terms of the (flat or Euclidean) Hilbert-Schmidt metric [11] [12, secs. 9.6-9.6, 14.3], and then, using the fundamentally important Bures (minimal monotone) metric [12, sec. 14.4] [13]. (In particular, they employed the Laguerre ensemble of random matrix theory [14] in both sets of computations (cf. [15]). The Bures and HS metrics were compared by Hall [16], who concluded that the Bures induced the "minimal-knowledge ensemble" (cf. [17]).) That is, we will seek now to extend the form of analysis applied in the Hilbert-Schmidt context in [1] to the Bures setting.

To begin, let us review the most elementary findings reported in [1, sec. II.A.1]. The simplest (four-parameter) scenario studied there posits a 4×4 density matrix ρ with fully general diagonal entries ($\rho_{11}, \rho_{22}, \rho_{33}, \rho_{44} = 1 - \rho_{11} - \rho_{22} - \rho_{33}$) and only one pair of real off-diagonal non-zero entries, $\rho_{23} = \rho_{32}$. The HS separability function for that scenario was

found to take the form [1, eq. (20)],

$$\mathcal{S}^{HS}_{[(2,3)]}(\mu) = \begin{cases} 2\mu & 0 \le \mu \le 1 \\ 2 & \mu > 1 \end{cases}, \tag{1}$$

where we will now primarily employ (purely as a matter of convenience) the variable $\mu = \sqrt{\frac{\rho_{11}\rho_{44}}{\rho_{22}\rho_{33}}}$, rather than $\nu = \mu^2$, as in [1, 18].

Allowing the 23- and 32-entries to be complex conjugates of one another, we further found for the corresponding separability function [1, eq. (22)]—where the wide tilde over an i, j pair will throughout indicate a complex entry (described by *two* parameters)—

$$\mathcal{S}^{HS}_{[\widetilde{(2,3)}]}(\mu) = (\sqrt{\frac{\pi}{2}}\mathcal{S}^{HS}_{[(2,3)]}(\mu))^2 = \begin{cases} \pi\mu^2 & 0 \le \mu \le 1 \\ \pi & \mu > 1 \end{cases}. \tag{2}$$

Further, permitting the 23- and 32-entries to be *quaternionic* conjugates of one another [19, 20], the corresponding separability function [1, eq. (24)]—where the wide hat over an i, j pair will throughout indicate a quaternionic entry (described by *four* parameters)—took the form

$$\mathcal{S}_{[\widehat{(2,3)}]}(\nu) = (\sqrt{\frac{\pi}{2}}\mathcal{S}^{HS}_{[\widetilde{(2,3)}]}(\mu))^2 = (\sqrt{\frac{\pi}{2}}\mathcal{S}^{HS}_{[(2,3)]}(\mu))^4 = \begin{cases} \frac{\pi^2\mu^4}{2} & 0 \le \mu \le 1 \\ \frac{\pi^2}{2} & \mu > 1 \end{cases}. \tag{3}$$

So, the real (1), complex (2), and quaternionic (3) HS separability functions accord perfectly with the Dyson index sequence $\beta = 1, 2, 4$ of random matrix theory [21]. "The value of β is given by the number of independent degrees of freedom per matrix element and is determined by the antiunitary symmetries ...It is a concept that originated in Random Matrix Theory and is important for the Cartan classification of symmetric spaces" [22, p. 480]. The Dyson index corresponds to the multiplicity of ordinary roots, in the terminology of symmetric spaces [23, Table 2]. However, we remain unaware of any specific line of argument using random matrix theory [14] that can be used to formally confirm the HS separability function Dyson-index-sequence phenomena we have noted above and observed in [1]. (The basic difficulty/novelty appears to be that the separability aspect of the problem introduces a totally new set of complicated constraints—*quartic* (biquadratic) in μ [1, eq. (5)] [18, eq. (7)]—that the multivariate integration must respect [1, sec. I.C].)

As a further recent exercise, unreported in [1], we found that setting any single one of the four components of the quaternionic entry, $x_{23} + \mathbf{i}y_{23} + \mathbf{j}ju_{23} + \mathbf{k}v_{23}$, in the scenario just described, to zero, yields the separability function,

$$
\mathcal{S}^{HS}_{[(2,\hat{3})]} = \begin{cases} \frac{4\pi\mu^3}{3} & 0 \leq \mu \leq 1 \\ \frac{4\pi}{3} & \mu > 1 \end{cases},
\tag{4}
$$

consistent, at least, in terms of the exponent of μ, with the Dyson-index pattern previously observed.

Continuing the analysis in [1], we computed the integrals

$$
V^{HS}_{sep/scenario} = \int_0^\infty \mathcal{S}^{HS}_{scenario}(\mu)\mathcal{J}^{HS}_{scenario}(\mu)d\mu,
\tag{5}
$$

of the products of these separability functions with the corresponding (univariate) marginal jacobian functions (which are obtained by integration over diagonal parameters only and *not* any of the off-diagonal x_{ij}'s and y_{ij}'s) for the reparameterization of ρ using the Bloore variables [1, eq. (17)]. This yielded the HS scenario-specific *separable* volumes $V^{HS}_{sep/scenario}$. The ratios of such separable volumes to the HS total volumes

$$
V^{HS}_{tot/scenario} = c^{HS}_{scenario} \int_0^\infty \mathcal{J}^{HS}_{scenario}(\mu)d\mu,
\tag{6}
$$

where $c^{HS}_{scenario}$ is a scenario-specific constant, gave us in [1] (invariably, it seems, exact) separability *probabilities*. (For the three scenarios listed above, these probabilities were, respectively, $\frac{3\pi}{16}, \frac{1}{3}$ and $\frac{1}{10}$.)

Based on the numerous scenario-specific analyses in [1], we are led to believe that the real, complex and quaternionic separability functions adhere to the Dyson-index pattern for general scenarios, when the Hilbert-Schmidt measure has been employed. This apparent adherence was of central importance in arriving at the assertions in [1, secs. IX.A.1 and IX.A.2] that the HS separability probabilities of generic [9-dimensional] real and [15-dimensional] complex two-qubit states are $\frac{8}{17}$ and $\frac{8}{33}$, respectively. There we had posited—using mutually supporting numerical and theoretical arguments—that [1, eq. (102)]

$$
\mathcal{S}_{real}(\mu) \propto \frac{1}{2}(3 - \mu^2)\mu,
\tag{7}
$$

and, further pursuing our basic Dyson-index ansätz (fitting our numerical simulation extremely well [1, Fig. 4]), that $(\mathcal{S}_{real}(\mu))^2 \propto \mathcal{S}_{complex}(\mu)$. (Obviously, we must as well make

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the further claim that $(\mathcal{S}_{real}(\mu))^4 \propto \mathcal{S}_{quat}(\mu)$. Unlike the real and complex cases, however, we have performed no numerical analyses such as those in [18] to guide us as to the proper coefficient of proportionality to employ. Thus, we have no specific assertion to advance as to the two-qubit quaternionic separability probability—although $\frac{8}{65}$ or $\frac{8}{129}$ might be readily suggested.)

Now, employing formulas (13) and (14) of Dittmann [24] for the *Bures* metric—which avoid the possibly problematical need for diagonalization of ρ—we were able to find the Bures volume elements for the same three basic (one pair of free off-diagonal entries) scenarios. We obtained for the real case,

$$dV^{Bures}_{[(2,3)]} = \frac{\sqrt{\rho_{11}}\sqrt{1-\rho_{11}-\rho_{22}}\sqrt{\rho_{22}}}{4\sqrt{1-x_{23}^2}\left(\rho_{22}\mu^2+\rho_{11}\right)\sqrt{\mu^2\rho_{22}^2+(1-\rho_{11})\rho_{11}}}d\rho_{11}d\rho_{22}dx_{23}d\mu, \qquad (8)$$

for the complex case,

$$dV^{Bures}_{[\widetilde{(2,3)}]} = \frac{\rho_{11}\rho_{22}\left(\rho_{11}+\rho_{22}-1\right)}{4\sqrt{1-y_{23}^2-x_{23}^2}\left(\rho_{22}\mu^2+\rho_{11}\right)\left(-\rho_{11}^2+\rho_{11}+\mu^2\rho_{22}^2\right)}d\rho_{11}d\rho_{22}dx_{23}dy_{23}d\mu, \qquad (9)$$

and for the quaternionic case,

$$dV^{Bures}_{[\widehat{(2,3)}]} = \frac{A}{B}d\rho_{11}d\rho_{22}dx_{23}dy_{23}du_{23}dv_{23}d\mu, \qquad (10)$$

where

$$A = -\rho_{11}^2\rho_{22}^2\left(\rho_{11}+\rho_{22}-1\right)^2,$$

and

$$B = 4\sqrt{1-u_{23}^2-v_{23}^2-x_{23}^2-y_{23}^2}\left(\rho_{22}\mu^2+\rho_{11}\right)\left(-\rho_{11}^2+\rho_{11}+\mu^2\rho_{22}^2\right)^2.$$

In analyzing the quaternionic case, we transformed—using standard procedures [19, p. 495] [25, eq. (17)]—the corresponding 4×4 density matrix into an 8×8 density matrix with (only) complex entries. To this, we found it most convenient to apply—since its eigenvalues and eigenvectors could be explicitly computed—the basic formula of Hübner [26] [27, p. 2664] for the Bures metric.

Integrating these three volume elements over all the (four, five or seven) variables, while enforcing nonnegative definiteness of ρ, we derived the Bures *total* (separable *and* nonseparable) volumes for the three scenarios—$V^{Bures}_{tot/[(2,3)]} = \frac{\pi^2}{12} \approx 0.822467$, $V^{Bures}_{tot/[\widetilde{(2,3)}]} = \frac{\pi^3}{64} \approx 0.484473$, and $V^{Bures}_{tot/[\widehat{(2,3)}]} = \frac{\pi^4}{768} \approx 0.126835$.

We note importantly that the Bures volume elements ((8), (9), (10)), in these three cases, can be *factored* into products of functions of the off-diagonal variables, u_{23}, v_{23}, x_{23}

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and y_{23}, and functions of the diagonal variables, ρ_{11}, ρ_{22} and μ. Now, we will integrate (one may transform to polar and spherical coordinates, as appropriate) just the factors —

$\frac{1}{\sqrt{1-x_{23}^2}}, \frac{1}{\sqrt{1-x_{23}^2-y_{23}^2}}$ and $\frac{1}{\sqrt{1-u_{23}^3-v_{23}^2-x_{23}^2-y_{23}^2}}$ —involving the off-diagonal variable(s) over those variables. In doing this, we will further enforce (using the recently-incorporated integration-over-implicitly-defined-regions feature of Mathematica) the Peres-Horodecki positive-partial-transpose-criterion [8, 9, 10], expressible as

$$\mu^2 - x_{23}^2 \geq 0 \tag{11}$$

in the real case,

$$\mu^2 - x_{23}^2 - y_{23}^2 \geq 0, \tag{12}$$

in the complex case, and

$$\mu^2 - x_{23}^2 - y_{23}^2 - u_{23}^2 - v_{23}^2 \geq 0, \tag{13}$$

in the quaternionic case. (None of the individual diagonal ρ_{ii}'s appears explicitly in these constraints, due to an attractive property of the Bloore [correlation coefficient/off-diagonal scaling] parameterization. Replacing μ^2 in these three constraints by simply unity, we obtain the non-negative definiteness constraints on ρ itself, which we also obviously must enforce.) Performing the indicated three integrations, we obtain the *Bures* separability functions,

$$\mathcal{S}_{[(2,3)]}^{Bures}(\mu) = \begin{cases} \pi & \mu \geq 1 \\ 2\sin^{-1}(\mu) & 0 < \mu < 1 \end{cases}, \tag{14}$$

$$\mathcal{S}_{[\widetilde{(2,3)}]}^{Bures}(\mu) = \begin{cases} 2\pi & \mu \geq 1 \\ 2\pi\left(1 - \sqrt{1-\mu^2}\right) & 0 < \mu < 1 \end{cases}, \tag{15}$$

and

$$\mathcal{S}_{[\widehat{(2,3)}]}^{Bures}(\mu) = \begin{cases} \frac{4\pi^2}{3} & \mu > 1 \\ \frac{2}{3}\pi^2\left(-\sqrt{1-\mu^2}\mu^2 - 2\sqrt{1-\mu^2} + 2\right) & 0 < \mu < 1 \end{cases}. \tag{16}$$

Then, utilizing these three separability functions—that is, integrating the products of the functions and the corresponding remaining *diagonal*-variable factors in the Bures volume elements ((8), (9)), ((10)) over the μ, ρ_{11} and ρ_{22} variables—we obtain *separable* volumes of $V_{sep/[(2,3)]}^{Bures} = 0.3658435525$ and

$$V_{sep/[\widetilde{(2,3)}]}^{Bures} = V_{tot/[\widetilde{(2,3)}]}^{Bures} - \frac{1}{32}\pi^2(-2C + \pi) = \frac{1}{64}\pi^2(4C - 6 + \pi) \approx 0.124211 \tag{17}$$
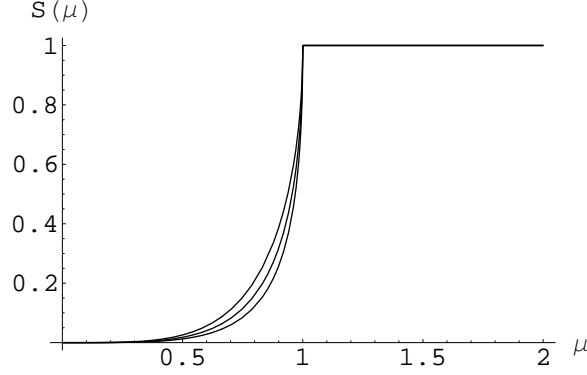
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FIG. 1: Joint plot of the normalized Bures *quaternionic* separability function $\frac{3\mathcal{S}^{Bures}_{\widehat{[(2,3)]}}(\mu)}{4\pi^2}$, the *square* of the normalized Bures *complex* separability function $\frac{\mathcal{S}^{Bures}_{\widetilde{[(2,3)]}}(\mu)}{2\pi}$, and the *fourth* power of the normalized Bures *real* separability function $\frac{\mathcal{S}^{Bures}_{[(2,3)]}(\mu)}{\pi}$. The order of dominance of the three curves is the same as the order in which they have been mentioned.

and consequent separability *probabilities*, respectively, of 0.4448124200 and (our only *exact* Bures separability probability result in this study (cf. [28])),

$$P^{Bures}_{sep/\widetilde{[(2,3)]}} = \frac{4C - 6 + \pi}{\pi} \approx 0.256384, \tag{18}$$

where $C \approx 0.915966$ is Catalan's constant (cf. [29]). (This constant appears commonly in estimates of combinatorial functions and in certain classes of sums and definite integrals [30, sec. 1.7].) Further, for the quaternionic case, $V^{Bures}_{sep/\widehat{[(2,3)]}} \approx 0.012954754466$, and $P^{Bures}_{sep/\widehat{[(2,3)]}} \approx 0.10213883862$. (The corresponding HS separability probability was also of the same relatively small magnitude, that is, $\frac{1}{10}$ [1, sec. II.A.3]. We have computed the various Bures separable volumes and probabilities to high numerical accuracy, hoping that such accuracy may be useful in searches for possible further exact formulas for them.)

So, the normalized—to equal 1 at $\mu = 1$—forms of these three separability functions are $\frac{\mathcal{S}^{Bures}_{[(2,3)]}(\mu)}{\pi}$, $\frac{\mathcal{S}^{Bures}_{\widetilde{[(2,3)]}}(\mu)}{2\pi}$ and $\frac{3\mathcal{S}^{Bures}_{\widehat{[(2,3)]}}(\mu)}{4\pi^2}$. In Fig. 1, we plot—motivated by the appearance of the Dyson indices in the analyses of [1]—the *fourth* power of the first (real) of these three normalized functions together with the *square* of the second (complex) function and the (untransformed) third (quaternionic) function itself. We find a very close,

$$\left(\frac{\mathcal{S}^{Bures}_{[(2,3)]}(\mu)}{\pi}\right)^4 \approx (\frac{\mathcal{S}^{Bures}_{\widetilde{[(2,3)]}}(\mu)}{2\pi})^2 \approx (\frac{3\mathcal{S}^{Bures}_{\widehat{[(2,3)]}}(\mu)}{4\pi^2}), \tag{19}$$

but now *not* exact fit, as we did find in [1] for their (also normalized) Hilbert-Schmidt counterparts $\frac{\mathcal{S}^{HS}_{[(2,3)]}(\mu)}{2}$, $\frac{\mathcal{S}^{HS}_{\widetilde{[(2,3)]}}(\mu)}{\pi}$ and $\frac{2\mathcal{S}^{HS}_{\widehat{[(2,3)]}}(\mu)}{\pi^2}$ ((1), (2), (3)).

As an additional exercise (cf. (4)), we have computed the Bures separability function in the case that a single one of the four components of the (2,3)-quaternionic entry is set to zero. Then, we have (falling into the same tight cluster in Fig. 1, when the $\frac{4}{3}$-power of its normalized form is plotted)

$$\mathcal{S}^{Bures}_{[(2,3)]} = \begin{cases} \frac{1}{8}\pi^2 \left(4 - \sqrt{2}\log\left(3 + 2\sqrt{2}\right)\right) & \mu > 1 \\ \frac{1}{4}\pi \left(\mu\sqrt{1-\mu^2} - \sin^{-1}(\mu)\right)\left(\sqrt{2}\log\left(3 + 2\sqrt{2} - 4\right)\right) & 0 < \mu < 1 \end{cases}. \quad (20)$$

We have been able, further, using the formulas of Dittmann [24], to compute the Bures volume elements for the corresponding (five-dimensional) real and (seven-dimensional) complex scenarios, in which *both* the $\{2,3\}$ and $\{1,2\}$ entries are allowed to freely vary. But these volume elements do not appear, now, to fully factorize into products of functions (as is the case for (8) and (9)) involving just $\rho_{11}, \rho_{22}, \mu$ and just the off-diagonal variables x_{ij}'s and y_{ij}'s. The requisite integrations are, then, more problematical and it seemed impossible to obtain a univariate separability function of μ.

For instance, in this regard, we have for the indicated five-dimensional real scenario that

$$dV^{Bures}_{[(1,2),(2,3)]} = \frac{1}{4}\sqrt{\frac{A}{BC(D+E)}}d\rho_{11}d\rho_{22}dx_{12}dx_{23}d\mu, \quad (21)$$

where

$$A = -\rho_{11}^2\rho_{22}^2\left(\rho_{11} + \rho_{22} - 1\right)\left(\left(\mu^2 - 1\right)\rho_{22} + 1\right), \quad (22)$$

$$B = \left(\rho_{22}\mu^2 + \rho_{11}\right)^2, C = x_{12}^2 + x_{23}^2 - 1, \quad (23)$$

$$D = \left(\rho_{11} + \rho_{22}\right)\left(x_{12}^2\rho_{22}\left(\rho_{22}\mu^2 + \rho_{11}\right)^2 - \left(\left(\mu^2 - 1\right)\rho_{22} + 1\right)\left(-\rho_{11}^2 + \rho_{11} + \mu^2\rho_{22}^2\right)\right) \quad (24)$$

and

$$E = -x_{23}^2\rho_{22}\left(\rho_{11} + \rho_{22} - 1\right)\left(-\rho_{11}^2 + \rho_{11} + \mu^2\rho_{22}^2\right). \quad (25)$$

So, no desired factorization is apparent.

However, the computational situation greatly improves if we let the (1,4) and (2,3)-entries be the two free ones. (These entries are the specific ones that are interchanged under the operation of partial transposition, so there is a greater evident symmetry in such a scenario.) Then, we found that the three Bures volume elements all do factorize into products of functions of off-diagonal entries and functions of diagonal entries. We have

$$dV^{Bures}_{[(1,4),(2,3)]} = \frac{1}{8}\sqrt{-\frac{1}{\left(x_{14}^2 - 1\right)\left(x_{23}^2 - 1\right)\left(\rho_{22} + \rho_{33} - 1\right)\left(\rho_{22} + \rho_{33}\right)}}d\rho_{11}d\rho_{22}d\rho_{33}dx_{14}dx_{23}, \quad (26)$$

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where simply for succinctness, we now show the volume elements before replacing the ρ_{33} variable by μ. (We note that the expression for $dV^{Bures}_{[(1,4),(2,3)]}$ is independent of ρ_{11}.) For the corresponding complex scenario,

$$dV^{Bures}_{\widetilde{[(1,4)},\widetilde{(2,3)]}} = \frac{1}{8}\sqrt{\frac{F}{G}}d\rho_{11}d\rho_{22}d\rho_{33}dr_{14}dr_{23}d\theta_{14}d\theta_{23}, \tag{27}$$

where

$$F = -r_{14}^2 r_{23}^2 \rho_{11}\rho_{22}\rho_{33}\left(\rho_{11}+\rho_{22}+\rho_{33}-1\right), \tag{28}$$

and

$$G = \left(r_{14}^2-1\right)\left(r_{23}^2-1\right)\left(\rho_{22}+\rho_{33}-1\right)^2\left(\rho_{22}+\rho_{33}\right)^2, \tag{29}$$

and we have now further shifted to polar coordinates, $x_{ij} + \mathbf{i}y_{ij} = r_{ij}(\cos\theta_{ij} + \mathbf{i}\sin\theta_{ij})$. For the quaternionic scenario, we have (using two sets of hyperspherical coordinates $(r_{14}, \theta_{14}^{(1)}, \theta_{14}^{(2)}, \theta_{14}^{(3)})$ and $(r_{23}, \theta_{23}^{(1)}, \theta_{23}^{(2)}, \theta_{23}^{(3)})$),

$$dV^{Bures}_{\widehat{[(1,4)},\widehat{(2,3)]}} = \frac{1}{8}\sqrt{\frac{\tilde{F}}{\tilde{G}}}d\rho_{11}d\rho_{22}d\rho_{33}dr_{14}dr_{23}d\theta_{14}^{(1)}d\theta_{14}^{(2)}d\theta_{14}^{(3)}d\theta_{23}^{(1)}d\theta_{23}^{(2)}d\theta_{23}^{(3)}, \tag{30}$$

where

$$\tilde{F} = \sin^2\left(\theta_{14}^{(1)}\right)\sin\left(\theta_{14}^{(2)}\right)\sin^2\left(\theta_{23}^{(1)}\right)\sin\left(\theta_{23}^{(2)}\right)r_{14}^3 r_{23}^3 \rho_{11}^{3/2}\rho_{22}^{3/2}\left(-\rho_{11}-\rho_{22}-\rho_{33}+1\right)^{3/2}\rho_{33}^{3/2} \tag{31}$$

and

$$\tilde{G} = \sqrt{1-r_{14}^2}\sqrt{1-r_{23}^2}\left(\rho_{22}+\rho_{33}-1\right)^2\left(\rho_{22}+\rho_{33}\right)^2. \tag{32}$$

The total Bures volume for the first (real) of these three scenarios is $V^{Bures}_{tot/[(1,4),(2,3)]} = \frac{\pi^3}{64} \approx 0.484473$, for the second (complex) scenario, $V^{Bures}_{tot/\widetilde{[(1,4)},\widetilde{(2,3)]}} = \frac{\pi^4}{192} \approx 0.507339$, and for the third (quaternionic), $V^{Bures}_{tot/\widehat{[(1,4)},\widehat{(2,3)]}} = \frac{\pi^6}{245760} \approx 0.0039119$.

In the two corresponding Hilbert-Schmidt (real and complex) analyses we have previously reported, we had the results [1, eq. (28)],

$$\mathcal{S}^{HS}_{[(1,4),(2,3)]}(\mu) = \begin{cases} 4\mu & 0 \le \mu \le 1 \\ \frac{4}{\mu} & \mu > 1 \end{cases}. \tag{33}$$

and [1, eq. (34)]

$$\mathcal{S}^{HS}_{\widetilde{[(1,4)},\widetilde{(2,3)]}}(\mu) = \begin{cases} \pi^2\mu^2 & 0 \le \mu \le 1 \\ \frac{\pi^2}{\mu^2} & \mu > 1 \end{cases}, \tag{34}$$

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thus, exhibiting the indicated exact (Dyson sequence) proportionality relation. We now found, for the two Bures analogs, that

$$
\mathcal{S}^{Bures}_{[(1,4),(2,3)]}(\mu) = \begin{cases} \pi^2 & \mu = 1 \\ 2\pi \csc^{-1}(\mu) & \mu > 1 \\ 2\pi \sin^{-1}(\mu) & 0 < \mu < 1 \end{cases} , \tag{35}
$$

$$
\mathcal{S}^{Bures}_{[\widetilde{(1,4)},\widetilde{(2,3)}]}(\mu) = \begin{cases} 16\pi^2 & \mu = 1 \\ 16\pi^2 \left(1 - \frac{\sqrt{\mu^2-1}}{\mu}\right) & \mu > 1 \\ 16\pi^2 \left(1 - \sqrt{1-\mu^2}\right) & 0 < \mu < 1 \end{cases} , \tag{36}
$$

and, further still, for the quaternionic scenario,

$$
\mathcal{S}^{Bures}_{[\widehat{(1,4)},\widehat{(2,3)}]}(\mu) = \begin{cases} \frac{16\pi^4}{9} & \mu = 1 \\ -\frac{8\pi^4\left(2\left(\sqrt{\mu^2-1}-\mu\right)\mu^2+\sqrt{\mu^2-1}\right)}{9\mu^3} & \mu > 1 \\ \frac{8}{9}\pi^4\left(-\sqrt{1-\mu^2}\mu^2 - 2\sqrt{1-\mu^2} + 2\right) & 0 < \mu < 1 \end{cases} . \tag{37}
$$

Employing these several results, we obtained that $V^{Bures}_{sep/[(1,4),(2,3)]} \approx 0.1473885131$, $V^{Bures}_{sep/[\widetilde{(1,4)},\widetilde{(2,3)}]} \approx 0.096915844$, and $V^{Bures}_{sep/[\widehat{(1,4)},\widehat{(2,3)}]} \approx 0.000471134100$ giving us real, complex and quaternionic separability probabilities of 0.3042243652, 0.19102778 and 0.120436049.

We see that for values of $\mu \in [0,1]$, the *normalized* forms of these three Bures separability functions are *identical* to the three obtained above ((14), (15), (16)) for the corresponding *single*-nonzero-entry scenarios. While those earlier functions were all constant for $\mu > 1$, we now have symmetrical behavior about $\mu = 1$ in the form, $\mathcal{S}^{Bures}_{scenario}(\mu) = \mathcal{S}^{Bures}_{scenario}(\frac{1}{\mu})$.

In Fig. 2, we show the analogous plot to Fig. 1, using the normalized (to equal 1 at $\mu = 1$) forms of the three additional Bures separability functions ((35), (36), (37)). We again, of course, observe a very close fit to the type of proportionality relations *exactly* observed in the Hilbert-Schmidt case ((33), (34)).

We were, further, able to compute the Bures volume element for the *three*-nonzero-entries complex scenario $[\widetilde{(1,2)}, \widetilde{(1,4)}, \widetilde{(2,3)}]$, but it was considerably more complicated in form than those reported above, so no additional analytical progress seemed possible.

Regarding the possible computation of Bures separability functions for the 9-dimensional real and 15-dimensional complex two-qubit states, we have found, preliminarily, that the
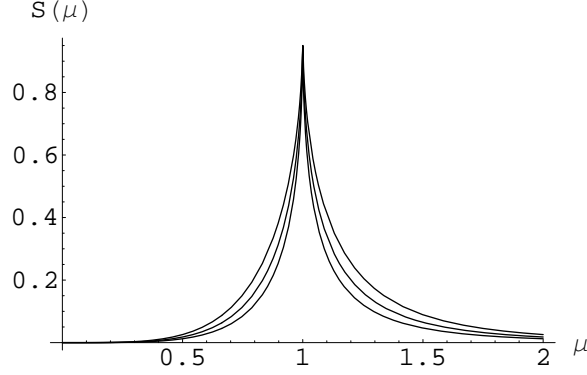
10

FIG. 2: Joint plot of the normalized Bures *quaternionic* separability function $\frac{9\mathcal{S}^{Bures}_{[(1,4),\widehat{(2,3)}]}(\mu)}{4}$, the *square* of the normalized Bures *complex* separability function $\frac{\mathcal{S}^{Bures}_{[\widetilde{(1,4)},\widetilde{(2,3)}]}(\mu)}{16\pi^2}$, and the *fourth* power of the normalized Bures *real* separability function $\frac{\mathcal{S}^{Bures}_{[(1,4),(2,3)]}(\mu)}{\pi^2}$. Over the interval $\mu \in [0,1]$, the three functions are identical—with the same order of dominance—to those in Fig. 1.

corresponding metric tensors (using the Bloore parameterization) decompose into 3×3 and 6×6, and 3×3 and 12×12 blocks, respectively. The 3×3 blocks themselves are identical in the two cases, and of precisely the (simple diagonal) form (if we employ hyperspherical coordinates) that Akhtarshenas found for the Bures metric using the coset parameterization [31, eq. (23)]. They, thus, depend only upon the diagonal entries (while in [31], the dependence, quite differently, was upon the eigenvalues). It appears, though, that the determinants—for which we presently lack succinct formulas—of the complementary 6×6 and 12×12 blocks, do depend upon all, diagonal and non-diagonal, parameters.

The close proximity observed in this study between certain separability results for the Hilbert-Schmidt and Bures metrics is perhaps somewhat similar in nature/explanation to a form of high similarity also observed in our previous analysis [32]. There, large scale numerical (quasi-Monte Carlo) analyses strongly suggested that the ratio of Hilbert-Schmidt separability probabilities of generic (rank-6) qubit-qutrit states (6×6 density matrices) to the separability probabilities of generically minimally degenerate (boundary/rank-5) qubit-qutrit states was equal to 2. (This has since been formally confirmed and generalized—in terms of positive-partial-transpose-ratios—to arbitrary bipartite systems by Szarek, Bengtsson and Życzkowski in [33]. They found that the set of positive-partial-transpose states is "pyramid decomposable" and, hence, is a body of constant height.) Parallel numerical ratio estimates also obtained in [32] based on the Bures (and a number of other monotone)

metrics were also surprisingly close to 2, as well (1.94334 in the Bures case [32, Table IX]). However, no exact value for the Bures qubit-qutrit ratio has ever been established, and our separability function results above, might be taken to suggest that the actual Bures ratio is not exactly equal to 2, but only quite close to it. (Possibly, in these regards, the Bures metric might profitably be considered as some perturbation of the flat Euclidean metric (cf. [34]).)

We plan to continue to study the forms the Bures separability functions take for qubit-qubit and qubit-qutrit scenarios, with the hope that we can achieve as much insight into the nature of Bures separability probabilities, if not more, than we obtained by examining the analogous Hilbert-Schmidt separability functions [1]. (In [35], we had formulated, based on extensive numerical evidence, conjectures—involving the silver mean, $\sqrt{2}-1$—for the Bures [and other monotone metric] separability probabilities of the 15-dimensional convex set of [complex] qubit-qubit states, which we would further aspire to test. One may also consider the use of monotone metrics other than the *minimal* Bures one [15]—such as the Kubo-Mori and Wigner-Yanase.) The analytical challenges to further progress, however, appear quite formidable.

Acknowledgments

[1] P. B. Slater, arXiv:0704.3723v2.

[2] F. J. Bloore, J. Phys. A **9**, 2059 (1976).

[3] H. Joe, J. Multiv. Anal. **97**, 2177 (2006).

[4] D. Kurowicka and R. Cooke, Lin. Alg. Applics. **372**, 225 (2003).

[5] D. Kurowicka and R. M. Cooke, Lin. Alg. Applics. **418**, 188 (2006).

[6] S. Guiasu, Phys. Rev. A **36**, 1971 (1987).

[7] V. E. Mkrtchian and V. O. Chaltykyan, Opt. Commun. **63**, 239 (1987).

[8] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[9] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[10] D. Bruß and C. Macchiavello, Found. Phys. **35**, 1921 (2005).

[11] K. Życzkowski and H.-J. Sommers, J. Phys. A **36**, 10115 (2003).

[12] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States* (Cambridge, Cambridge, 2006).

[13] H.-J. Sommers and K. Życzkowski, J. Phys. A **36**, 10083 (2003).

[14] M. L. Mehta, *Random Matrices* (Elsevier/Academic, Amsterdam, 2004).

[15] A. Andai, J. Phys. A **39**, 13641 (2006).

[16] M. J. W. Hall, Phys. Lett. A **242**, 123 (1998).

[17] P. B. Slater, Phys. Lett. A. **247**, 1 (1998).

[18] P. B. Slater, Phys. Rev. A **75**, 032326 (2007).

[19] S. L. Adler, *Quaternionic quantum mechanics and quantum fields* (Oxform, New York, 1995).

[20] M. Asorey, G. Scolarici, and L. Solombrino, Phys. Rev. A **76**, 012111 (2007).

[21] F. J. Dyson, Commun. Math. Phys. **19**, 235 (1970).

[22] J. B. Kogut, M. A. Stephanov, D.Toublan, J. J. M. Verbaarschot, and A. Zhitnitsky, Nucl. Phys. B **582**, 477 (2000).

[23] M. Caselle and U. Magnea, Phys. Rep. **394**, 41 (2004).

[24] J. Dittmann, J. Phys. A **32**, 2663 (1999).

[25] P. B. Slater, J. Math. Phys. **37**, 2682 (1996).

[26] M. Hübner, Phys. Lett. A **163**, 239 (1992).

[27] J. Dittmann, Sem. Sophus Lie **3**, 73 (1993).

[28] P. B. Slater, Euro. Phys. J. B **17**, 471 (2000).

[29] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002).

[30] S. R. Finch, *Mathematical Constants* (Cambridge, New York, 2003).

[31] S. J. Akhtarshenas, arXiv:0705.1965.

[32] P. B. Slater, Phys. Rev. A **71**, 052319 (2005).

[33] S. Szarek, I. Bengtsson, and K. Życzkowski, J. Phys. A **39**, L119 (2006).

[34] D. J. Gross, M. J. Perry, and L. G. Yaffe, Phys. Rev. A **25**, 330 (1982).

[35] P. B. Slater, J. Geom. Phys. **53**, 74 (2005).